

Fig. 6. Characteristic impedances of four-conductor line in a square shield  
vers  $w/h$  and  $t/h$  at  $\epsilon_r = \mu_r = 1$ ,  $s/h = 0.2$  and  $d/h = 0.2$ .

and  $Z^{ee}$  of the four-conductor line with a square shield ( $B=1$ ), are presented graphically in Fig. 5—for the case  $2s+w=b$ ,  $2d+t=h$  and in Fig. 6—when  $s/h=0.2$ ,  $d/h=0.2$ . The comparison of data for the value of the impedances  $Z^{eo}$  and  $Z^{oo}$  from Fig. 5 with the corresponding results calculated from Getsinger's graphs [7] gives agreement within 2–3 percent.

The numerical data for the pair of the characteristic impedances  $Z^{eo}$ ,  $Z^{oo}$  and  $Z^{ee}$ ,  $Z^{oo}$  can be used for the design of coupled transmission lines in square shield [8]. The graphs for the other lines shown in Figs. 3 and 4 are presented in papers [4]–[6].

#### REFERENCES

- [1] S. A. Ivanov, "On the design of the characteristic impedance of strip transmission lines by the self-consistent field method," *Compt. Rend. Acad. Bulg. Sci.*, vol. 29, pp. 819–822, 1976.
- [2] S. A. Ivanov and V. R. Vionov, "Calculation of the asymmetrical stripline characteristic impedance by self-consistent field method," *Int. J. Electron.*, vol. 46, pp. 81–89, Jan. 1979.
- [3] J. W. Duncan, "Characteristic impedances of multiconductor strip transmission lines," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-13, pp. 107–118, Jan. 1965.
- [4] S. A. Ivanov, "Calculation of the characteristic impedances of the shielded coupled stripline by the self-consistent field method," *Bulg. J. Phys.*, vol. IV, pp. 324–334, 1977.
- [5] S. A. Ivanov and R. I. Tchonchev, "Study on coupled strip transmission line with front connection between conductors," *Bulg. J. Phys.*, vol. V, pp.

422–430, 1978.

- [6] S. A. Ivanov, "Design of the unbalanced strip transmission line by the self-consistent field method," *Bulg. J. Phys.*, vol. IV, pp. 203–212, 1977.
- [7] W. J. Getsinger, "Coupled rectangular bars between parallel plates," *IRE Trans. Microwave Theory Tech.*, vol. MTT-10, pp. 65–72, 1962.
- [8] S. A. Ivanov, "Calculation of the characteristic impedances of coupled strip lines in rectangular shield," *Compt. Rend. Acad. Bulg. Sci.*, vol. 30, pp. 519–522, 1977.

#### On the Orthogonality of Approximate Waveguide Mode Functions

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**Abstract**—For many waveguides, only approximate solutions for the mode functions are available and in such cases the question arises, whether the orthogonality property of the exact modes can be preserved. This problem is addressed in the present paper. A fairly general method of

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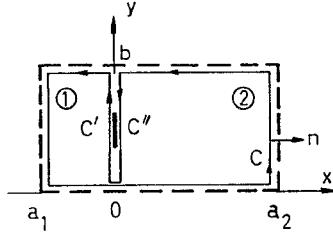


Fig. 1. Unit cell waveguide with conducting strip, occurring in a periodic dipole array antenna.

solution is considered and it is shown that in spite of two consecutive approximations the resultant mode functions are indeed orthogonal. Examples that have been analyzed include a rectangular waveguide with a septum, a rectangular waveguide with an axial, conducting strip and a (phased array) unit cell waveguide with one or more axial, conducting strips.

## INTRODUCTION

For many waveguides of practical interest, an orthogonal set of mode functions is known to exist, but no closed form solutions are available. For such cases approximate solutions have to be used. A general problem in this context is whether the orthogonality property of the exact modes is preserved. The orthogonality, which is essential in order to expand an arbitrary field into a set of modes, may well be lost in the course of approximations. This problem, which seems not to have been discussed before, is addressed in the present paper.

Clearly, an approximate solution always depends on the manner in which it is derived. However, since the present method is fairly general and can be applied to a large class of waveguides, the effort to establish the orthogonality of the resultant approximate solutions for the modes is justified and worthwhile.

## DERIVATION OF THE APPROXIMATE SOLUTIONS

We shall use the method given in [1] to derive approximate solutions for the waveguide modes. This method is based upon a decomposition of the waveguide cross section into subregions, in each of which Helmholtz's equation is separable. The field expressions of each region are then matched, in Galerkin's sense, across the common boundaries.

As an illustrative example, we consider a unit cell waveguide [2] with an axial strip conductor (Fig. 1). This structure is useful in the analysis of a periodic phased array with dipole elements, where the region between the plane of the dipoles and the groundplane can be treated as a unit cell with an axial conductor, representing the dipole support [3].

The  $E$ -modes can be derived from a mode potential  $\psi(x, y)$  which satisfies the 2-dimensional Helmholtz equation.

$$(\nabla^2 + \lambda)\psi(x, y) = 0 \quad (1)$$

with the boundary conditions

$$\psi(a_2, y) = \psi(a_1, y) e^{i\alpha} \quad (2a)$$

$$\partial\psi(a_2, y)/\partial x = e^{i\alpha} \partial\psi(a_1, y)/\partial x \quad (2b)$$

$$\psi(x, b) = \psi(x, 0) e^{i\beta} \quad (3a)$$

$$\partial\psi(x, b)/\partial y = e^{i\beta} \partial\psi(x, 0)/\partial y \quad (3b)$$

$$\psi = 0 \text{ on strip} \quad (4)$$

where  $\alpha$  and  $\beta$  are the imposed phase shifts in the  $x$ - and  $y$ -direction, across the unit cell. In view of the  $z$ -directed current on the strip,  $\partial\psi/\partial x$  will be discontinuous across the strip. Setting

$\psi = \psi_1$  in region 1 ( $x < 0$ ) and  $\psi = \psi_2$  in region 2 ( $x > 0$ ) we have

$$\psi_1(-0, y) = \psi_2(+0, y) \quad (5)$$

$$\partial\psi_1(-0, y)/\partial x = \partial\psi_2(+0, y)/\partial x = \rho(y) \quad (6)$$

where  $\rho(y)$  corresponds to the (as yet) unknown strip current.

Separating the coordinates of (1) and enforcing (3) leads to

$$\Psi_p(x, y) = \sum_1^{\infty} f_{pn}(x) g_n(y), \quad p = 1, 2 \quad (7)$$

where

$$f_{pn}(x) = C_{pn} \exp(ik_{xn}x) + D_{pn} \exp(-ik_{xn}x)$$

$$g_n(y) = \exp(ik_{yn}y) \quad (8)$$

$$k_{yn} = (\beta + n2\pi)/b, \quad k_{xn}^2 = \lambda - k_{yn}^2.$$

The boundary conditions (2a), (2b), (5), and (6) provide four equations from which the four unknown sets of expansion coefficients  $C_{pn}$ ,  $D_{pn}$  can be determined in terms of  $\rho$  and  $\lambda$ .

Applying the Ritz-Galerkin method, we now expand  $\rho$  in a finite number of basis functions  $\{e_m\}^M$  with unknown coefficients

$$\rho(y) \simeq \begin{cases} \sum_1^M A_m e_m(y), & \text{on strip} \\ 0, & \text{outside strip} \end{cases} \quad (9)$$

This introduces the first approximation. The second approximation is committed when, for the purpose of numerical evaluation, the infinite series (7) is truncated, leading to

$$\tilde{\psi}_p(x, y) = \sum_1^N f_{pn}(x) g_n(y). \quad (10)$$

Finally, requiring that the approximate potential  $\tilde{\psi}(\rho)$  has the same projection as the exact potential in the space spanned by  $\{e_m\}^M$  gives in view of (4)

$$\int_{\text{strip}} \tilde{\psi} e_m^* dy = 0, \quad m = 1, \dots, M. \quad (11)$$

Equation (11) represents a set of  $M$  linear homogeneous equations for the unknowns  $\{A_m\}^M$  and thus for nontrivial solutions the determinant of the coefficient matrix must vanish. This condition represents a dispersion relation from which the infinite set of eigenvalues  $\{\lambda_{\mu}\}_{\mu=1}^{\infty}$  is determined, and for each  $\lambda_{\mu}$  an amplitude vector  $\{A_m(\mu)\}_{m=1}^M$ , and a corresponding  $\tilde{\psi}(x, y, \mu)$  and mode function  $\nabla \tilde{\psi}(x, y, \mu)$  are obtained.

## ORTHOGONALITY OF THE APPROXIMATE SOLUTIONS

In order to establish the orthogonality of the set  $\{\nabla \tilde{\psi}(\mu)\}_{\mu=1}^{\infty}$ , we use Green's theorem giving

$$\begin{aligned} & (\lambda_{\mu} - \lambda_{\nu}^*) \int \int_S \nabla \tilde{\psi}(\mu) \cdot \nabla \tilde{\psi}(\nu)^* dS \\ & = \int_C \left[ \lambda_{\mu} \tilde{\psi}(\mu) \frac{\partial \tilde{\psi}(\nu)^*}{\partial n} - \lambda_{\nu}^* \tilde{\psi}(\nu)^* \frac{\partial \tilde{\psi}(\mu)}{\partial n} \right] ds \end{aligned} \quad (12)$$

where  $S$  is the unit cell cross section,  $C$  the contour shown in Fig. 1, and  $\partial/\partial n$  denotes the normal derivative.

In view of the periodic boundary conditions (2), (3), contributions to the contour integral from the unit cell walls cancel, and we are left with the integral over  $C''$  and  $C'$ . This reduces to

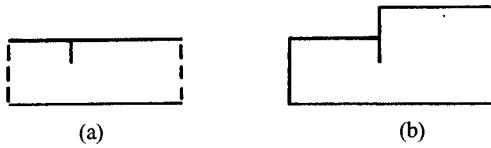


Fig. 2. (a) Unit cell in parallel plate region with septa. (b) Waveguide with septum.

two similar integrals, the first one given by

$$\lambda_\mu \int_0^b \tilde{\Psi}(\mu) \left[ \frac{\partial \tilde{\Psi}(\nu)}{\partial x} \Big|_{x=-0} - \frac{\partial \tilde{\Psi}(\nu)}{\partial x} \Big|_{x=+0} \right]^* dy \quad (13)$$

and the second integral obtained by interchanging  $\mu$  and  $\nu$  and complex conjugating.

The above integral can be shown to vanish by considering

$$I = \int_{\text{strip}} \tilde{\Psi}(\mu) \left[ \frac{\partial \tilde{\Psi}(\nu)}{\partial x} \Big|_{x=-0} - \frac{\partial \tilde{\Psi}(\nu)}{\partial x} \Big|_{x=+0} \right]^* dy. \quad (14)$$

Substituting (6) and (9) in (14) leads, in view of (11) to  $I=0$ . Alternatively, if in (14) the series expansions for  $\tilde{\Psi}(\mu)$  and  $\partial \tilde{\Psi}(\nu)/\partial x$  are substituted we obtain, due to the orthogonality of the functions  $g_n(y)$

$$I=0 = \int_0^b \tilde{\Psi}(\mu) \left[ \frac{\partial \tilde{\Psi}(\nu)}{\partial x} \Big|_{x=-0} - \frac{\partial \tilde{\Psi}(\nu)}{\partial x} \Big|_{x=+0} \right]^* dy. \quad (15)$$

Therefore, the integral (13) vanishes and the orthogonality of the approximate  $E$ -mode functions  $\nabla \tilde{\Psi}(\nu)$  is proved.

For the case of TEM-modes (1) is reduced to Laplace's equation and boundary condition (4) is changed to  $\psi = \text{constant}$  on strip.

For  $H$ -modes,  $\psi$  is related to the  $z$ -component of the magnetic field and the boundary conditions are:

(4)  $\partial \psi / \partial x = 0$  on strip

(5)  $\partial \psi(-0) / \partial x = \partial \psi(+0) / \partial x$  and

(6)  $\psi(-0) - \psi(+0) = \rho$

where  $\rho$  represents the  $y$ -directed surface current on the strip. In a manner similar to the above, the approximate solutions are then

derived and the orthogonality of the TEM-,  $E$ -, and,  $H$ -mode approximations established.

Further examples in which we have found the same orthogonality include a unit cell waveguide with two conducting strips, a unit cell in a parallel plate waveguide with septa (Fig. 2a), a regular waveguide with a conducting strip and a waveguide with a septum (Fig. 2b). In the last case, since the waveguide height is different in regions 1 and 2, two different sets of  $g_n$ -functions, i.e.,  $\{g_{1n}(y)\}$  and  $\{g_{2n}(y)\}$  are required. However, a sufficient condition for mode orthogonality is that  $\{g_{1n}\}$  be an orthogonal set on  $(0, b_1)$  and  $\{g_{2n}\}$  be orthogonal on  $(0, b_2)$ . The Sturm-Liouville operator ensures this property of the sets on their respective intervals.

## CONCLUSION

As a result, we arrive at the conclusion that the approximate solutions for the modes have the same orthogonality property as the exact solutions. This is a desirable but by no means obvious result, since two consecutive approximations were made. The solution is based upon Galerkin's method, which uses equal basis and testing functions [4]. Had these functions been unequal, orthogonality would not have been preserved.

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## REFERENCES

- [1] J.P. Montgomery, "On the complete eigenvalue solution of ridged waveguide," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-19, pp. 547-555, June 1971.
- [2] *Microwave Scanning Antennas*, R.C. Hansen, Ed., vol. 11, New York: Academic, 1966.
- [3] E. Mayer and A. Hessel, "Feed region modes in dipole phased arrays," to be published.
- [4] R. Harrington, *Field Computation by Moment Methods*. New York: Macmillan, 1968.